Chapter 9: Equations

Equation 9.1:

$$\mathrm{COV}\left(\varepsilon_i, \varepsilon_j\right) \neq 0$$

Equation 9.2:

$$\mathrm{V}(b) = \mathrm{V}\left(\frac{\sum_{i=1}^{n}\left(x_i - \overline{x}\right) y_i}{\sum_{i=1}^{n}\left(x_i - \overline{x}\right)^2}\right)$$
$$= \frac{\mathrm{V}\left(\sum_{i=1}^{n}\left(x_i - \overline{x}\right) y_i\right)}{\left(\sum_{i=1}^{n}\left(x_i - \overline{x}\right)^2\right)^2}$$

Equation 9.3:

$$\mathrm{V}(b) = \frac{\sum_{i=1}^{n}\mathrm{V}\left(\left(x_i - \overline{x}\right) y_i\right)}{\left(\sum_{i=1}^{n}\left(x_i - \overline{x}\right)^2\right)^2} + 2\frac{\sum_{i=2}^{n}\sum_{j=1}^{i-1}\mathrm{COV}\left(\left(x_i - \overline{x}\right) y_i, \left(x_j - \overline{x}\right) y_j\right)}{\left(\sum_{i=1}^{n}\left(x_i - \overline{x}\right)^2\right)^2}$$

Equation 9.4:

$$\mathrm{COV}\left(k_i y_i, k_j y_j\right) = k_i k_j \mathrm{COV}\left(y_i, y_j\right)$$

Equation 9.5:

$$2\frac{\sum_{i=2}^{n}\sum_{j=1}^{i-1}\left(x_i-\overline{x}\right)\left(x_j-\overline{x}\right)\mathrm{COV}\left(\varepsilon_i,\varepsilon_j\right)}{\left(\sum_{i=1}^{n}\left(x_i-\overline{x}\right)^2\right)^2}$$

Equation 9.6:

$$\mathrm{V}\left(b\right)=\frac{\sigma^2}{\sum_{i=1}^{n}\left(x_i-\overline{x}\right)^2}+2\frac{\sum_{i=2}^{n}\sum_{j=1}^{i-1}\left(x_i-\overline{x}\right)\left(x_j-\overline{x}\right)\mathrm{COV}\left(\varepsilon_i,\varepsilon_j\right)}{\left(\sum_{i=1}^{n}\left(x_i-\overline{x}\right)^2\right)^2}$$

Equation 9.7:

$$2\frac{\sum_{i=2}^{n}\sum_{j=1}^{i-1}\left(x_i-\overline{x}\right)\left(x_j-\overline{x}\right)e_ie_j}{\left(\sum_{i=1}^{n}\left(x_i-\overline{x}\right)^2\right)^2}$$

Equation 9.8:

$$2\frac{\sum_{i=2}^{n}\sum_{j=\max(1,i-L)}^{i-1}\left(1-\frac{i-j}{L+1}\right)\left(x_i-\overline{x}\right)\left(x_j-\overline{x}\right)e_ie_j}{\left(\sum_{i=1}^{n}\left(x_i-\overline{x}\right)^2\right)^2}$$

Equation 9.9:

$$\begin{gathered}y_i=-15,725+4,009.8x_i\\ \left(3,842.3\right)\ \ \left(288.40\right)\end{gathered}$$

Equation 9.10:

$$\varepsilon_i = \rho\varepsilon_{i-1} + \nu_i$$

Equation 9.11:

$$\mathrm{E}\left(\nu_i\right) = 0,\ \mathrm{V}\left(\nu_i\right) = \sigma^2,\ \mathrm{COV}\left(\nu_i, \nu_j\right) = 0$$

Equation 9.12:

$$\varepsilon_{i-1} = \rho\varepsilon_{i-2} + \nu_{i-1}$$

Equation 9.13:

$$\varepsilon_i = \rho\left(\rho\varepsilon_{i-2} + \nu_{i-1}\right) + \nu_i = \rho^2\varepsilon_{i-2} + \rho\nu_{i-1} + \nu_i$$

Equation 9.14:

$$\varepsilon_i = \rho^{i+\infty}\varepsilon_{-\infty} + \sum_{j=0}^{i+\infty}\rho^j\nu_{i-j}$$

Equation 9.15:

$$\varepsilon_i \approx \sum_{j=0}^{\infty}\rho^j\nu_{i-j}$$

Equation 9.16:

$$\mathrm{E}\left(\varepsilon_i\right) \approx \mathrm{E}\left(\sum_{j=0}^{\infty}\rho^j\nu_{i-j}\right) = \sum_{j=0}^{\infty}\mathrm{E}\left(\rho^j\nu_{i-j}\right) = \sum_{j=0}^{\infty}\rho^j\mathrm{E}\left(\nu_{i-j}\right) = \sum_{j=0}^{\infty}\rho^j\left(0\right) = 0$$

Equation 9.17:

$$\mathrm{V}\left(\varepsilon_i\right) \approx \sum_{j=0}^{\infty} \mathrm{V}\left(\rho^j \nu_{i-j}\right) = \sum_{j=0}^{\infty} \rho^{2j} \mathrm{V}\left(\nu_{i-j}\right) = \sum_{j=0}^{\infty} \rho^{2j} \left(\sigma^2\right) = \sigma^2 \sum_{j=0}^{\infty} \rho^{2j}$$

Equation 9.18:

$$\left(1-\rho^2\right)\sum_{j=0}^{\infty} \rho^{2j} = \sum_{j=0}^{\infty} \rho^{2j} - \sum_{j=0}^{\infty} \rho^2 \rho^{2j} = \sum_{j=0}^{\infty} \rho^{2j} - \sum_{j=0}^{\infty} \rho^{2(1+j)} = 1-\rho^{2\infty+2} \approx 1$$

Equation 9.19:

$$\mathrm{V}\left(\varepsilon_i\right) \approx \frac{\sigma^2}{1-\rho^2}$$

Equation 9.20:

$$\mathrm{COV}\left(\varepsilon_i, \varepsilon_{i-1}\right) = \mathrm{E}\left(\left(\varepsilon_i - \mathrm{E}(\varepsilon_i)\right)\left(\varepsilon_{i-1} - \mathrm{E}\left(\varepsilon_{i-1}\right)\right)\right)$$

Equation 9.21:

$$\mathrm{COV}\left(\varepsilon_i, \varepsilon_{i-1}\right) = \mathrm{E}\left(\rho\varepsilon_{i-1}^2 + \nu_i \varepsilon_{i-1}\right) = \mathrm{E}\left(\rho\varepsilon_{i-1}^2\right) + \mathrm{E}\left(\nu_i \varepsilon_{i-1}\right)$$

Equation 9.22:

$$\mathrm{COV}\left(\varepsilon_i, \varepsilon_{i-1}\right) = \rho\ \mathrm{E}\left(\varepsilon_{i-1}^2\right) + \mathrm{E}\left(\nu_i \varepsilon_{i-1}\right)$$

Equation 9.23:

$$\rho \, \mathrm{E}\left(\varepsilon_{i-1}^2\right) = \rho \, \mathrm{V}\left(\varepsilon_{i-1}\right) \approx \frac{\rho\sigma^2}{1-\rho^2}$$

Equation 9.24:

$$\mathrm{E}\left(\nu_i \varepsilon_{i-1}\right) = \mathrm{E}\left(\sum_{j=0}^{\infty} \rho^j \nu_i \nu_{i-1-j}\right) = \sum_{j=0}^{\infty} \rho^j \mathrm{E}\left(\nu_i \nu_{i-1-j}\right)$$

Equation 9.25:

$$\mathrm{COV}\left(\varepsilon_i, \varepsilon_{i-1}\right) = \rho \, \mathrm{E}\left(\varepsilon_{i-1}^2\right) = \frac{\rho\sigma^2}{1-\rho^2}$$

Equation 9.26:

$$\mathrm{CORR}\left(\varepsilon_i, \varepsilon_{i-1}\right) = \frac{\mathrm{COV}\left(\varepsilon_i, \varepsilon_{i-1}\right)}{\left(+\sqrt{\mathrm{V}\left(\varepsilon_i\right)}\right)\left(+\sqrt{\mathrm{V}\left(\varepsilon_{i-1}\right)}\right)} = \frac{\frac{\rho\sigma^2}{1-\rho^2}}{\sqrt{\frac{\sigma^2}{1-\rho^2}}\sqrt{\frac{\sigma^2}{1-\rho^2}}} = \rho$$

Equation 9.27:

$$\mathrm{COV}\left(\varepsilon_i, \varepsilon_{i-2}\right) = \frac{\rho^2\sigma^2}{1-\rho^2}$$

Equation 9.28:

$$\mathrm{COV}\left(\varepsilon_i, \varepsilon_{i-j}\right) = \frac{\rho^j\sigma^2}{1-\rho^2}$$

Equation 9.29:

$$y_i = \alpha + \beta x_i + \varepsilon_i = \alpha + \beta x_i + \rho \varepsilon_{i-1} + \nu_i$$

Equation 9.30:

$$\rho y_{i-1} = \rho\alpha + \rho\beta x_{i-1} + \rho\varepsilon_{i-1}$$

Equation 9.31:

$$\left(\alpha - \rho\alpha\right) + \beta\left(x_i - \rho x_{i-1}\right) + \nu_i$$

Equation 9.32:

$$y_i - \rho y_{i-1}$$

Equation 9.33:

$$y_i - \rho y_{i-1} = \left(\alpha - \rho\alpha\right) + \beta\left(x_i - \rho x_{i-1}\right) + \nu_i$$

Equation 9.34:

$$y_i - \rho y_{i-1} = \tilde{y}_i$$

Equation 9.35:

$$x_i - \rho x_{i-1} = \tilde{x}_i$$

Equation 9.36:

$$\alpha - \rho\alpha = \tilde{\alpha}$$

Equation 9.37:

$$\tilde{y}_i = \tilde{\alpha} + \beta\tilde{x}_i + \nu_i$$

Equation 9.38:

$$b_{\text{GLS}} = \frac{\sum_{i=2}^{n}\left(\tilde{x}_i - \bar{\tilde{x}}\right)\tilde{y}_i}{\sum_{i=2}^{n}\left(\tilde{x}_i - \bar{\tilde{x}}\right)\tilde{x}_i}$$

Equation 9.39:

$$\bar{\tilde{x}} = \frac{\sum_{i=2}^{n}\tilde{x}_i}{n}$$

Equation 9.40:

$$\bar{\tilde{y}} = \frac{\sum_{i=2}^{n}\tilde{y}_i}{n}$$

Equation 9.41:

$$\hat{\rho} = \text{CORR}\left(e_i, e_{i-1}\right) = \frac{\text{COV}\left(e_i, e_{i-1}\right)}{\text{SD}\left(e_i\right)\text{SD}\left(e_{i-1}\right)}$$

Equation 9.42:

$$\mathrm{COV}\left(e_i, e_{i-1}\right) = \frac{\sum_{i=2}^{n}\left(e_i - \bar{e}\right)\left(e_{i-1} - \bar{e}\right)}{n-2} \approx \frac{\sum_{i=2}^{n} e_i e_{i-1}}{n-2}$$

Equation 9.43:

$$\hat{\rho} \approx \frac{\frac{\sum_{i=2}^{n}\left(e_i e_{i-1}\right)}{n-2}}{\sqrt{\frac{\sum_{i=1}^{n} e_i^2}{n-1}}\sqrt{\frac{\sum_{i=2}^{n} e_{i-1}^2}{n-2}}} \approx \frac{\sum_{i=2}^{n} e_i e_{i-1}}{\sqrt{\sum_{i=1}^{n} e_i^2}\sqrt{\sum_{i=2}^{n} e_{i-1}^2}}$$

Equation 9.44:

$$\mathrm{DW} = \frac{\sum_{i=2}^{n}\left(e_i - e_{i-1}\right)^2}{\sum_{i=1}^{n} e_i^2} = \frac{\sum_{i=2}^{n} e_i^2}{\sum_{i=1}^{n} e_i^2} + \frac{\sum_{i=2}^{n} e_{i-1}^2}{\sum_{i=1}^{n} e_i^2} - 2\frac{\sum_{i=2}^{n} e_i e_{i-1}}{\sum_{i=1}^{n} e_i^2}$$

Equation 9.45:

$$\mathrm{DW} \approx 2\left(1 - \hat{\rho}\right)$$

Equation 9.46:

$$\hat{\rho} = 1 - \frac{\mathrm{DW}}{2}$$

Equation 9.47:

$$\tilde{y}_i = \tilde{a}_{\text{GLS}} + b_{\text{GLS}}\tilde{x}_i + e_i$$

Equation 9.48:

$$b_{\text{GLS}} = \frac{\sum_{i=1}^{n}\left(\tilde{y}_i - \bar{\tilde{y}}\right)\tilde{x}_i}{\sum_{i=1}^{n}\left(\tilde{x}_i - \bar{\tilde{x}}\right)\tilde{x}_i}$$

Equation 9.49:

$$\tilde{a}_{\text{GLS}} = \bar{\tilde{y}} - b_{\text{GLS}}\bar{\tilde{x}}$$

Equation 9.50:

$$a_{\text{GLS}} = \frac{\tilde{a}_{\text{GLS}}}{1-\hat{\rho}}$$

Equation 9.51:

$$\begin{array}{cc} y_i = -14,569 + & 3,934x_i \\ (4,017) & (86.66) \end{array}$$

Equation 9.52:

$$\varepsilon_i = \gamma\varepsilon_{i-4} + \nu_i$$